Interactive Distributed Source Coding for Network Function Computation

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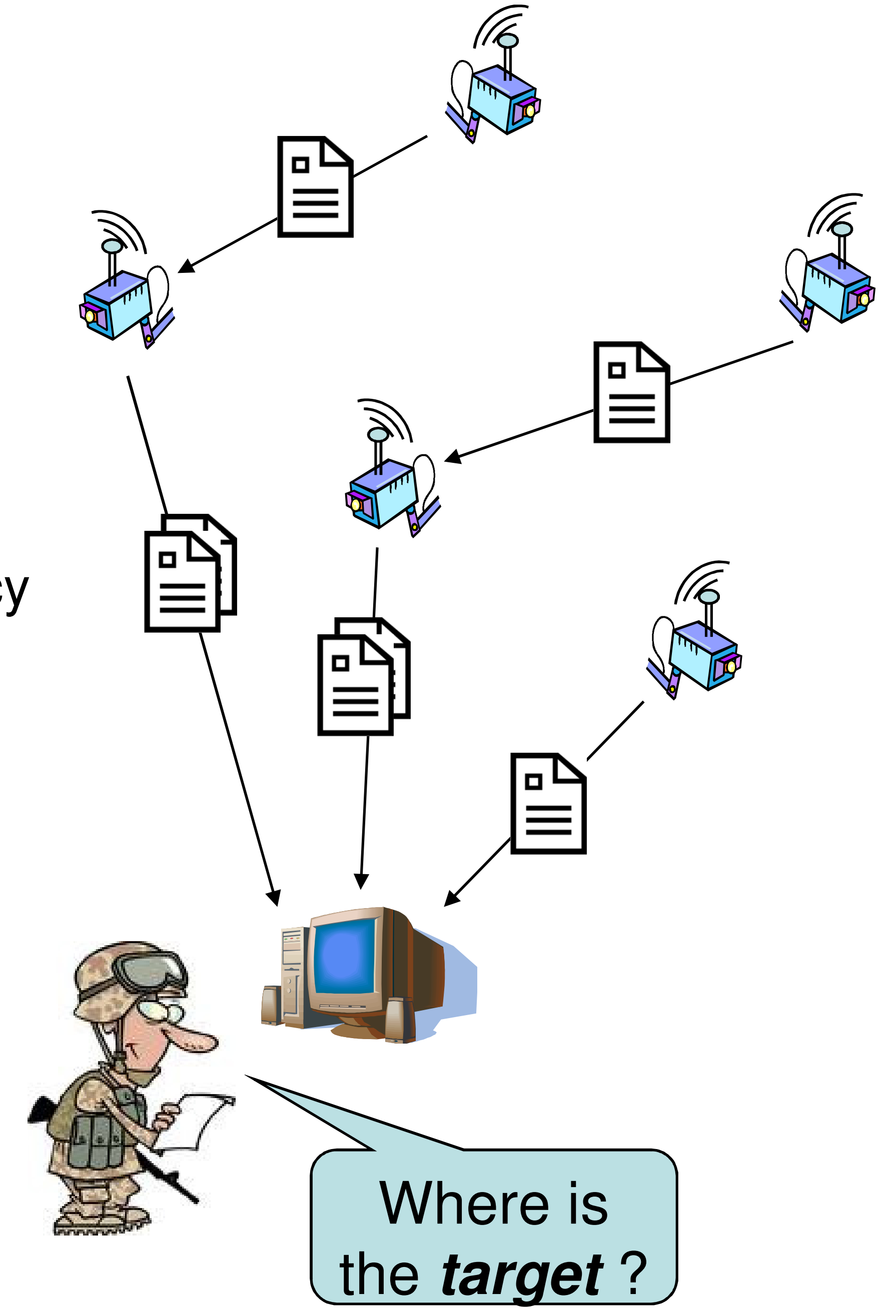
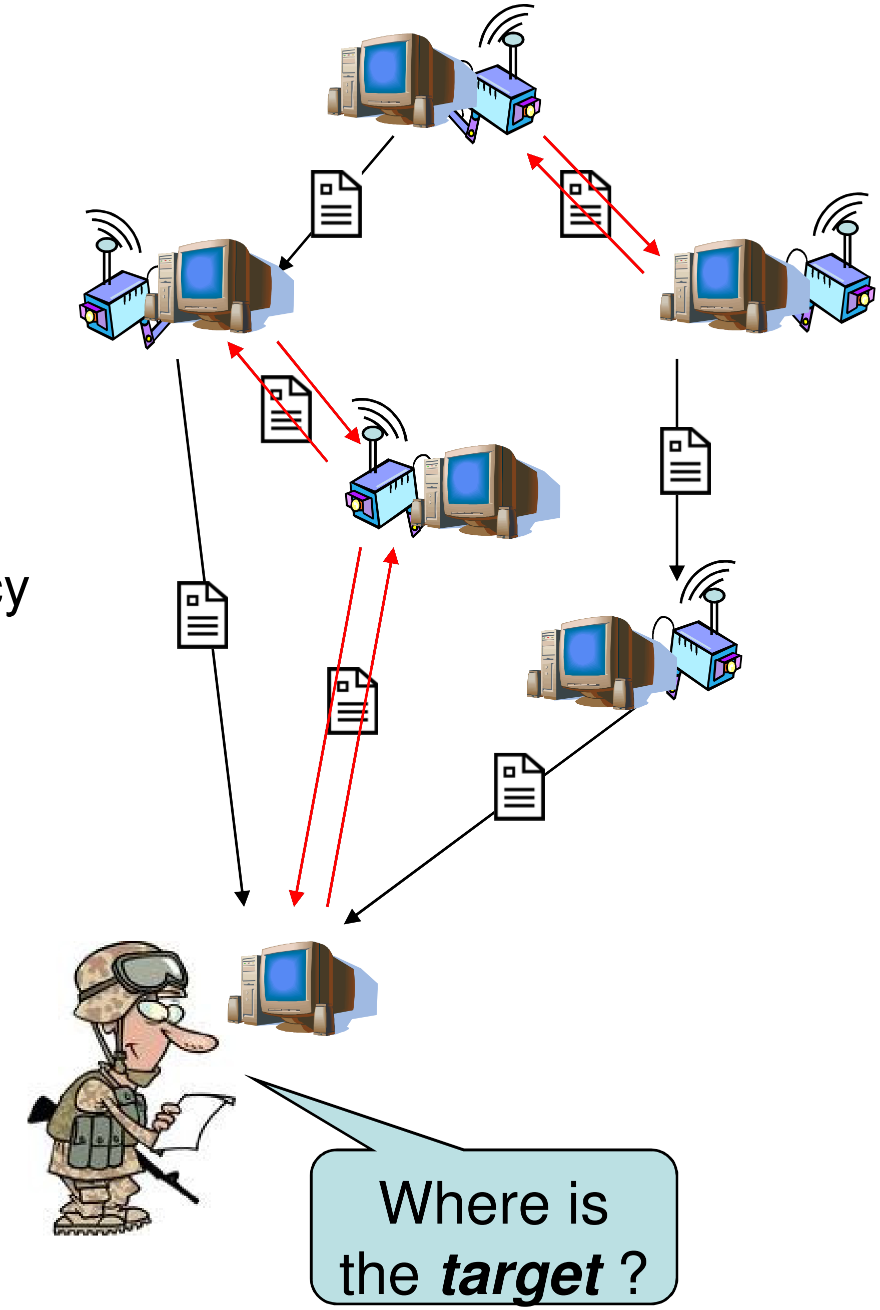
Motivation

- Sensor networks:
  - Don't need entire data, only functions
- Traditional data networks:
  - Move entire data to destination
  - Centralized computing
  - Inefficient
  - Resources: blocklength, SNR, frequency



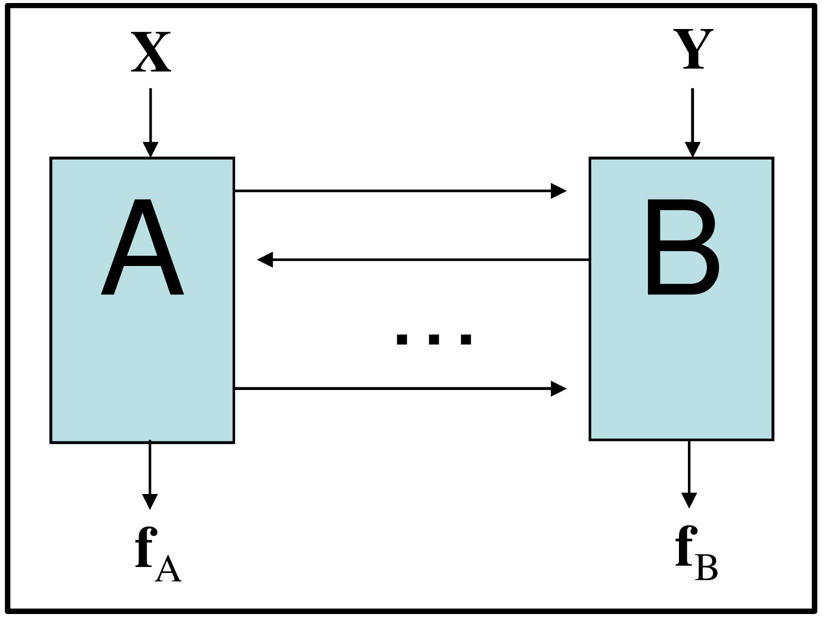
Motivation

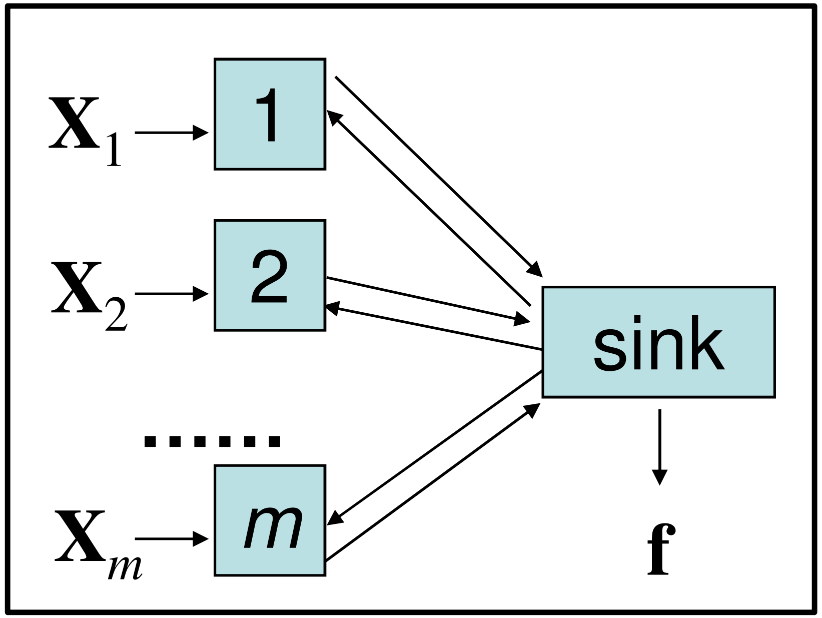
- Sensor networks:
  - Don’t need entire data, only functions
- Traditional data networks:
  - Move entire data to destination
  - Centralized computing
  - Inefficient
  - Resources: blocklength, SNR, frequency
- In-network distributed computing:
  - Process data as it moves
  - Efficient
  - New resource: two-way multi-round interaction



Motivation

- Interactive function computation problems:
  - Distributed source coding framework
  - Infinite-message interaction
  - Goal: characterize the ultimate limits of interaction





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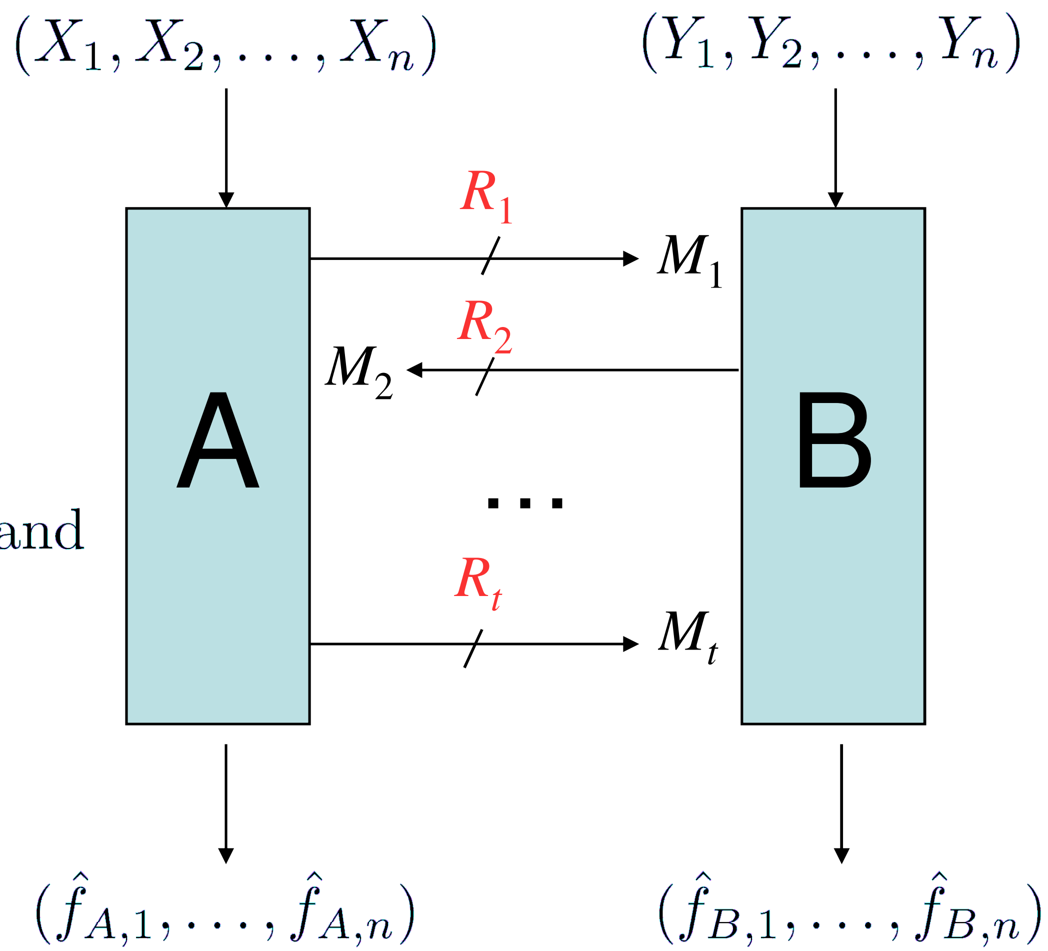
Outline

- Focus on 2-terminal interactive function computation problem
  - New functional characterization of infinite-message min sum-rate
  - Iterative algorithm for computing infinite-message min sum-rate
  - 2 messages can strictly improve the Wyner-Ziv R-D function
- Extension to multi-terminal problems
  - Star networks: interaction can change the scaling law
  - Collocated networks: new bounds order-wise better than cut-set bounds

2-terminal interactive function computation

- n samples $(X_i, Y_i) \sim \text{iid } p_{XY}$
- Samplewise function computation at A and B
- t alternating messages
- $(R_1,\ldots,R_t)$ is admissible if there exists a sequence of codes: as $n \to \infty, \quad (\#\text{ bits msg } j)/n \to R_j$ and $\Pr[\text{comp. error}] \to 0$ (lossless, can extend to lossy)
- Minimum sum-rate:

$$R^A_{sum,t} = \min \sum_{i=1}^{t} R_t$$



need $(f_A(X_1, Y_1), \ldots, f_A(X_n, Y_n)), \quad (f_B(X_1, Y_1), \ldots, f_B(X_n, Y_n))$

Goal:

Characterize and compute

$$R_{sum,\infty} := \lim_{t\to\infty} R^A_{sum,t}$$

- Understand ultimate benefit of cooperative interaction
- "Unexplored" dimension for asymptotic analysis:
  - (possibly) **infinite messages with infinitesimal rate**

Related work

- Communication complexity [Yao, Orlitsky,…]
  - Focus on Pr(comp. error) = 0
  - Usually about bits not rate
- Two-way source coding [Kaspi IT'85]
  - Source reproduction
- Coding for computing [Orlitsky & Roche IT'01]
  - Two messages

$R_{sum,t}$ for finite t : Solved

Single-letter characterization [Ma, Ishwar: ISIT'08, arXiv Nov'08]

$$R^A_{sum,t} = \min_{U^t}[I(X;U^t|Y) + I(Y;U^t|X)]$$

constraints

Markov chains

$$U_i - (X, U^{i-1}) - Y, \ i \text{ odd}$$
$$U_i - (Y, U^{i-1}) - X, \ i \text{ even}$$

Entropy constraints

$$H(f_A(X,Y)|X,U^t) = 0$$
$$H(f_B(X,Y)|Y,U^t) = 0$$

Cardinality bounds

$$|\mathcal{U}_j| \le \begin{cases} |\mathcal{X}| \left(\prod_{i=1}^{j-1} |\mathcal{U}_i|\right) + t - j + 3, & j \text{ odd}, \\ |\mathcal{Y}| \left(\prod_{i=1}^{j-1} |\mathcal{U}_i|\right) + t - j + 3, & j \text{ even}. \end{cases}$$

- Achievability: sequence of Wyner-Ziv-like codes
- Finite dimensional optimization problem

How to compute $R_{sum,\infty}$?

- Idea 1:
  - Pick a large t, compute $R^A_{sum,t}$, pray this is a good approximation
    - ☺ Finite-dimensional optimization problem
    - ☹ How large t ?
    - ☹ Dimension explodes exponentially with t
- Idea 2:
  - Compute $R^A_{sum,t}$ for $t = 1, 2, \ldots$ till change is "negligible"
    - ☺ Finite-dimensional optimization problem (for each t)
    - ☹ Multiple problems, solve from scratch
    - ☹ Dimension explodes exponentially with t
- All this effort only for one p_{XY}
- Any hope?

A new approach

- View $R_{sum,\infty}(p_{XY})$ as a functional of p_{XY}

- New convex-geometric characterization of $R_{sum,\infty}(p_{XY})$
  - For entire functional $R_{sum,\infty}(\bullet)$ (not for only one fixed p_{XY})
  - Provides optimality test for admissible sum-rate functionals
  - Family of lower bounds for $R_{sum,\infty}(\bullet)$

- Alternating "convexification" algorithm for $R_{sum,\infty}$
  - Each iteration uses "same amount" of computation
  - Reuses results from previous steps
  - Works with the entire $R_{sum,\infty}(p_{XY})$ "surface"

A new approach

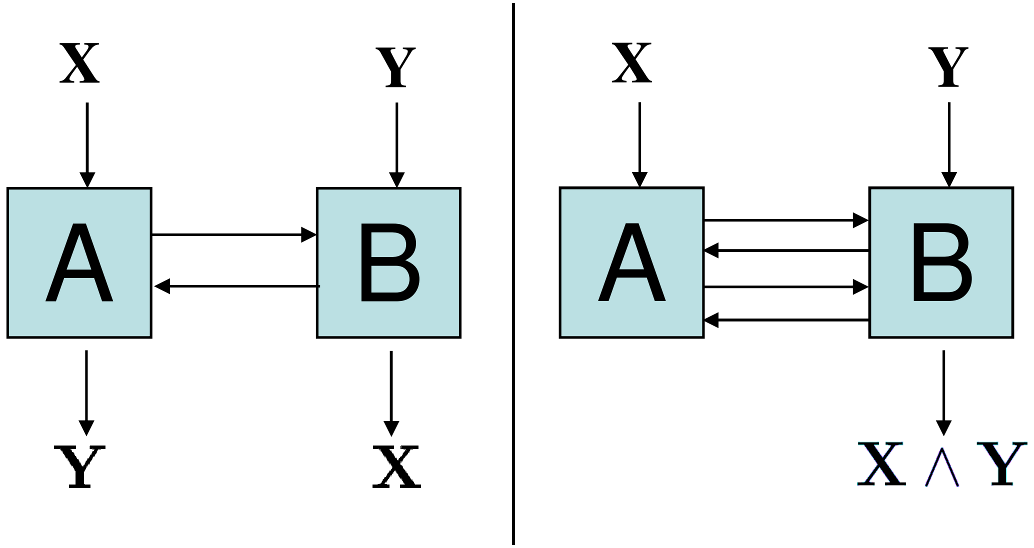
Rest of this talk:

- Illustrate new approach through one simple example for 2-terminal lossless function computation
- Extension to general lossy two-terminal computation
  - Focus: benefit of interaction for lossy source reproduction
- Extension to multi-terminal problems (brief)

Example: compute AND at terminal B

- Sources: $X \perp\!\!\!\perp Y, X \sim \text{Ber}(p), Y \sim \text{Ber}(q)$
- Only B computes: $f_A(X,Y) = 0, \quad f_B(X,Y) = X \wedge Y$ (AND)
- Goal: Characterize $R_{sum,\infty}(p, q)$ as a function of (p, q)
- Rate reduction functional:

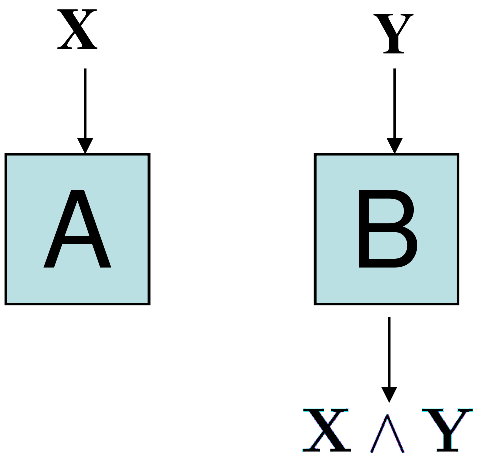
$$\rho_t^A := H(X|Y) + H(Y|X) \quad - \quad R_{sum,t}^A$$



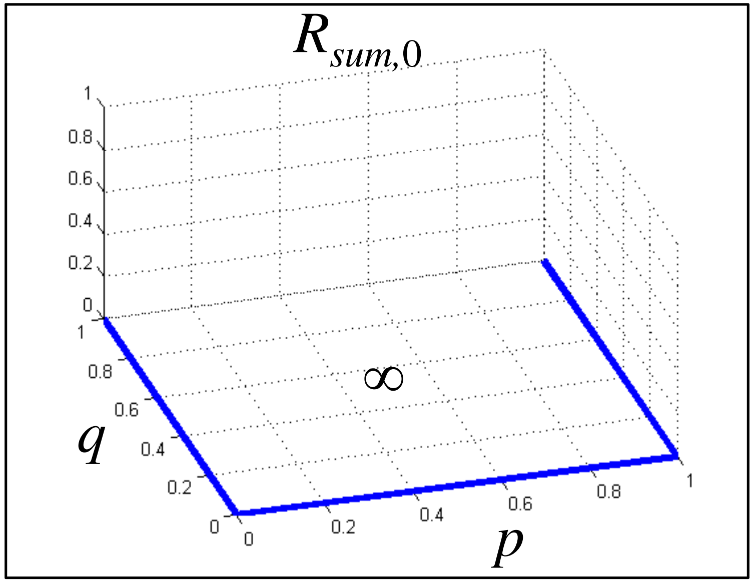
- Find $R_{sum,\infty} \iff$ find $\rho_\infty := H(X|Y) + H(Y|X) - R_{sum,\infty}$

Example: compute AND at terminal B

- Consider $t = 0$:
  - Feasible for special boundary distributions
  - If infeasible, define rate := ∞



$$R_{sum,0}(p,q) = \begin{cases} 0, & \text{if } p = 0 \text{ or } q = 0 \quad (X \wedge Y = 0) \\ & \text{or } p = 1 \quad (X \wedge Y = Y) \\ \infty, & \text{otherwise} \quad (X \wedge Y \text{ not determined}) \end{cases}$$



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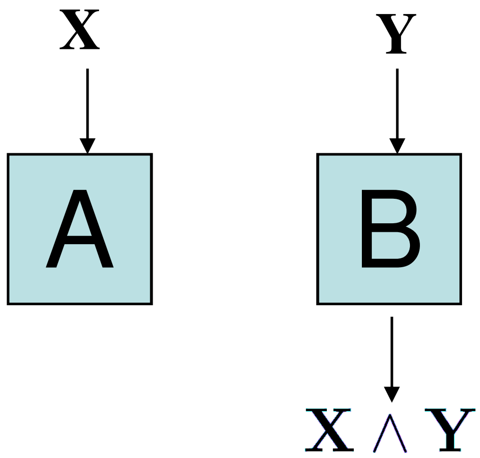

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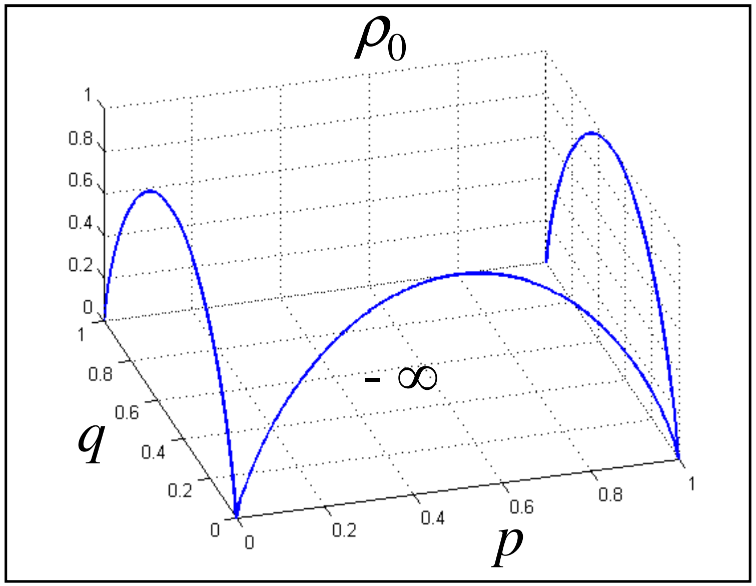
$$\rho_0(p,q) = \begin{cases} h(p) + h(q), & \text{if } p = 0 \text{ or } q = 0 \\ & \text{or } p = 1 \\ -\infty, & \text{otherwise} \end{cases}$$



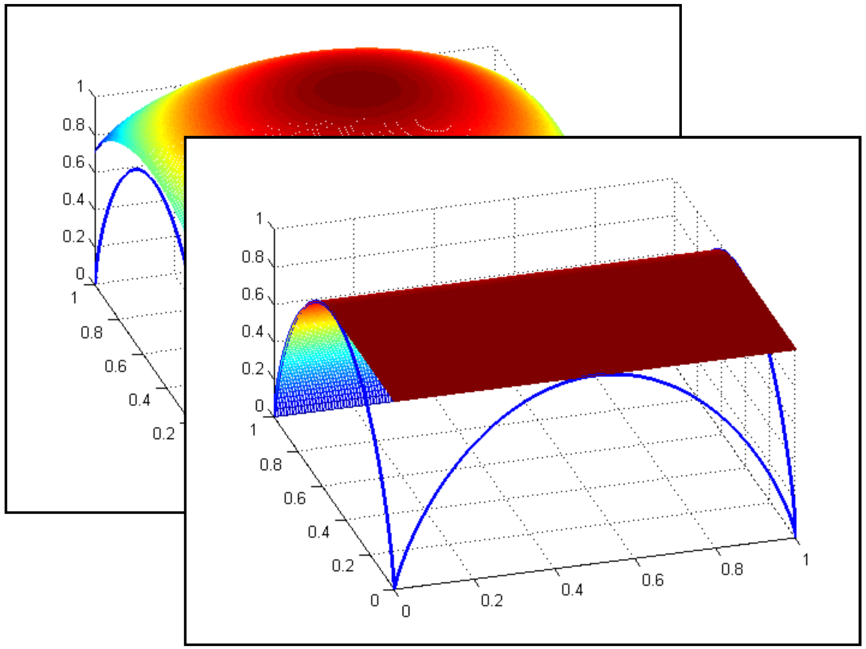
New characterization of ρ_∞

"Limit-free" characterization of ρ_∞ [Ma, Ishwar: Allerton 09]

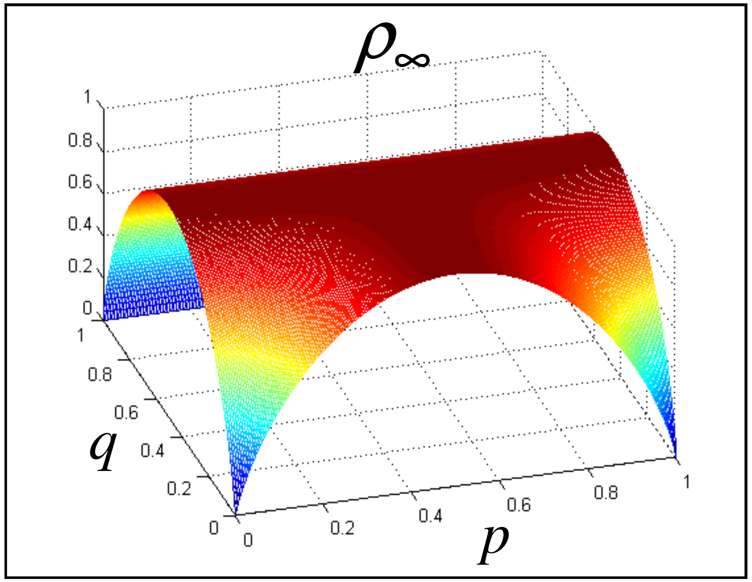
ρ_∞ is the least element of $\mathcal{F}$, where

$$\mathcal{F} := \left\{ \rho(p,q) \;\middle|\; \begin{array}{ll} 1. & \text{For all } (p,q),\ \rho(p,q) \geq \rho_0(p,q) \\ 2. & \text{For all } q,\ \rho(p,q) \text{ is concave w.r.t. } p \\ 3. & \text{For all } p,\ \rho(p,q) \text{ is concave w.r.t. } q \end{array} \right\}$$

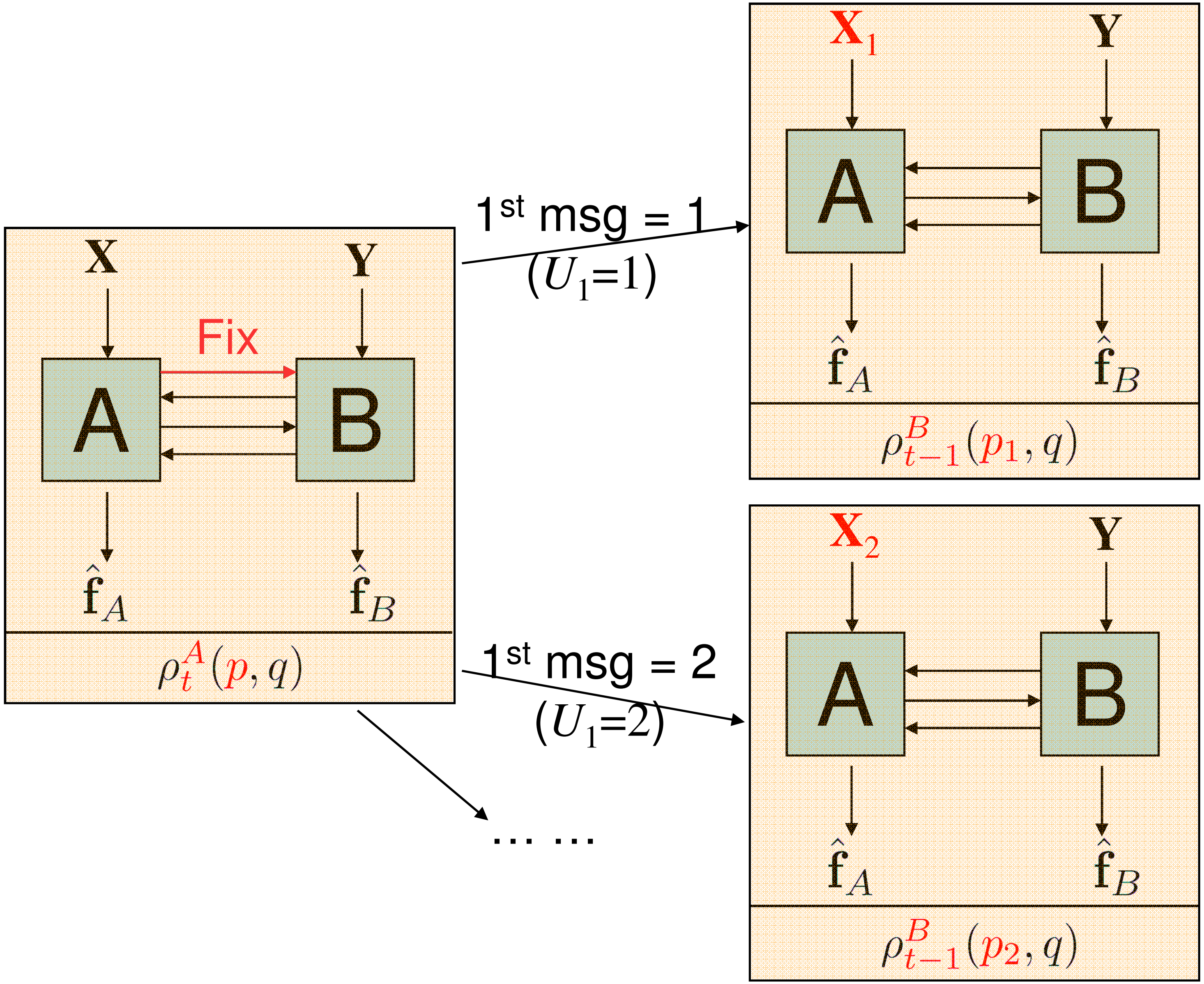
$\mathcal{F} = \{$



$\cdots \}$



Key insight: the subproblem viewpoint

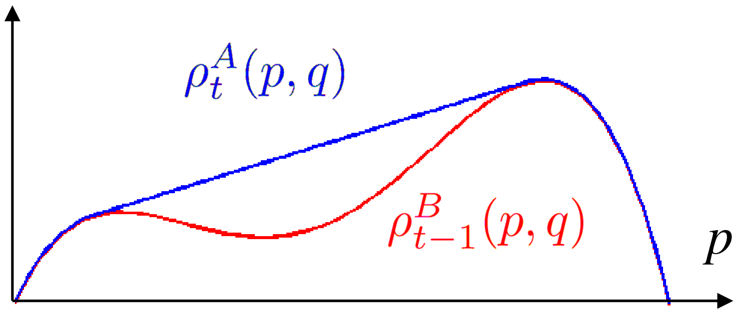


Connection between ρ_t^A and ρ_{t-1}^B ?

Subproblem viewpoint (continued)

$$\begin{aligned} R^A_{sum,t} &= \min_{U^t}[I(X;U^t|Y) + I(Y;U^t|X)] \\ \Rightarrow \rho^A_t &= \max_{U^t}[H(X|Y,U^t_2,U_1) + H(Y|X,U^t_2,U_1)] \\ &= \max_{U_1}\left[\sum_{u_1} p_{U_1}(u_1)\rho^B_{t-1}(p_{u_1},q)\right] \end{aligned}$$

- $\rho^A_t(p,q) = \max[$ convex combination of $\rho^B_{t-1}(p_1,q), \rho^B_{t-1}(p_2,q), \ldots]$

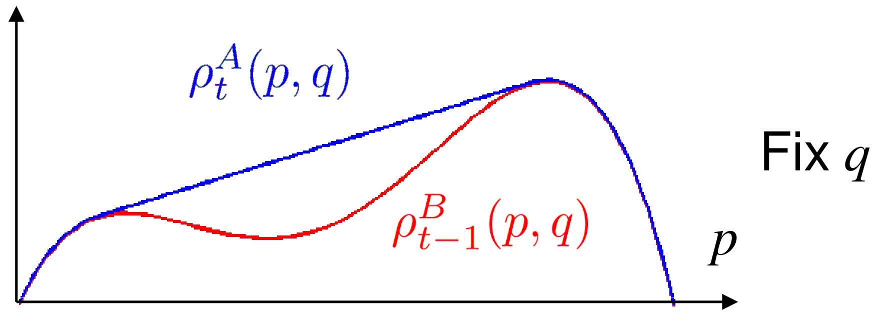


Fix q

"Concavify"

  – $\rho^A_t =$ the smallest concave function above ρ^B_{t-1}
  – $\mathrm{hypo}(\rho^A_t) =$ convex hull of $\mathrm{hypo}(\rho^B_{t-1})$

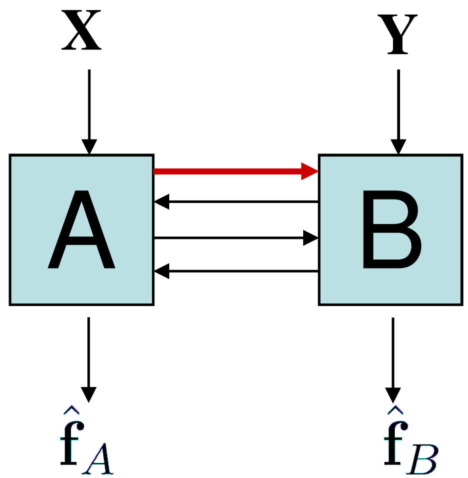
Subproblem viewpoint (continued)



- ρ_t^A concave in p, ρ_t^B concave in q

- ρ_{t-1}^B not concave in p $\Leftrightarrow$ $\rho_t^A > \rho_{t-1}^B$

  $\Leftrightarrow$ beneficial to add a message A $\rightarrow$ B



- ρ_∞ : not beneficial to add any message

  $\Leftrightarrow$ concave in p and q, respectively

Closed form expression of ρ_∞

"Limit-free" characterization of ρ_∞ [Ma, Ishwar: Allerton 09]

ρ_∞ is the least element of $\mathcal{F}$, where

$$\mathcal{F} := \left\{ \rho(p,q) \;\middle|\; \begin{array}{ll} 1. & \text{For all } (p,q),\ \rho(p,q) \geq \rho_0(p,q) \\ 2. & \text{For all } q,\ \rho(p,q) \text{ is concave w.r.t. } p \\ 3. & \text{For all } p,\ \rho(p,q) \text{ is concave w.r.t. } q \end{array} \right\}$$

- Admissible sum-rate R^*: (method in [Ma, Ishwar: ISIT'08])

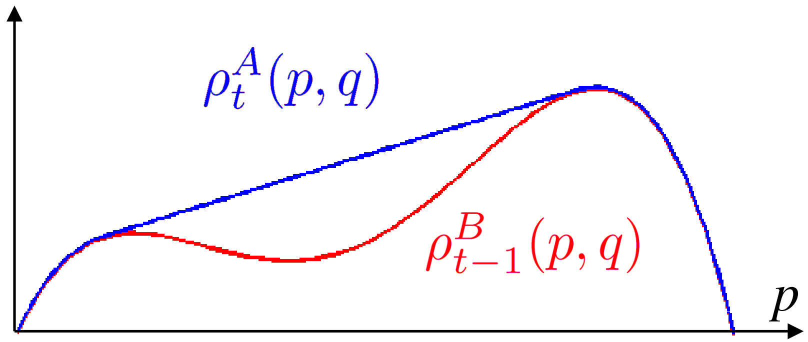
$$R^*(p,q) = \begin{cases} h_2(p) + ph_2(q) + p\log_2 q + p(1-2q)\log_2 e, & \text{if } 0 \leq p \leq q \leq 1/2, \\ R^*(q,p), & \text{if } 0 \leq q \leq p \leq 1/2, \\ R^*(1-p,q), & \text{if } 0 \leq q \leq 1/2 \leq p \leq 1, \\ h_2(p), & \text{if } 1/2 \leq q \leq 1. \end{cases}$$

- Each ρ in $\mathcal{F}$ is a UB of ρ_∞ $\Rightarrow$ $h(p) + h(q) - \rho$ is a LB of $R_{sum,\infty}$
- Optimality Test: Verify $\rho^* := h(p) + h(q) - R^* \in \mathcal{F} \Rightarrow R^* \leq R_{sum,\infty}$
- Therefore $R^* = R_{sum,\infty}$

Alternating concavification algorithm

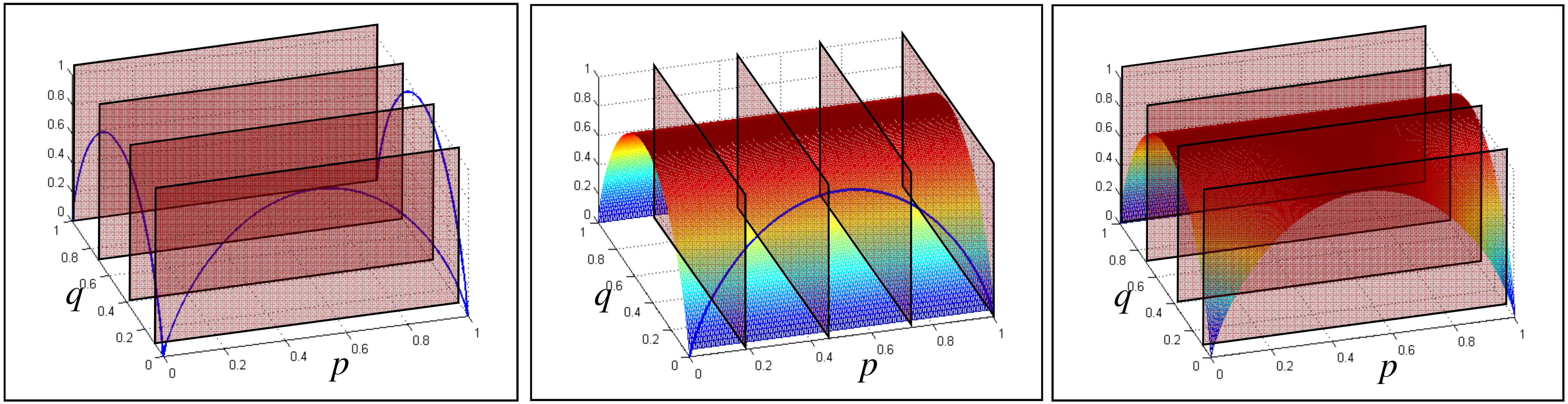
- Recall

$$\rho^B_{t-1}(p,q) \xrightarrow[\text{Fix } q]{\text{Concavify wrt } p} \rho^A_t(p,q)$$



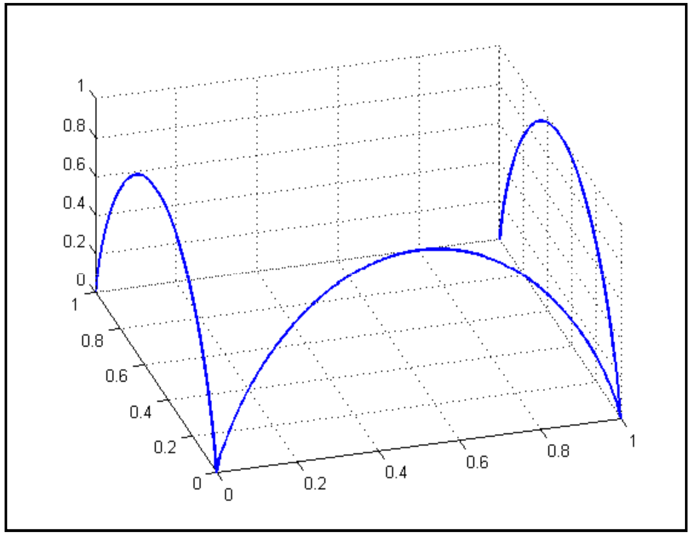
- Iterative algorithm

$$\rho_0(p,q) \xrightarrow[\text{Fix } q]{\text{Concavify wrt } p} \rho^A_1(p,q) \xrightarrow[\text{Fix } p]{\text{Concavify wrt } q} \rho^B_2(p,q) \longrightarrow \cdots$$

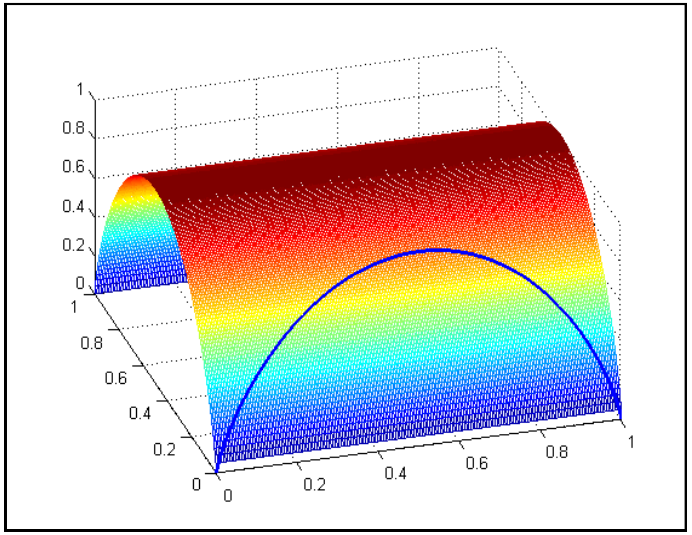


- Each iteration: "same amount" of computation
- Obtain the whole surface $\rho_t(p,q)$ for all (p,q)

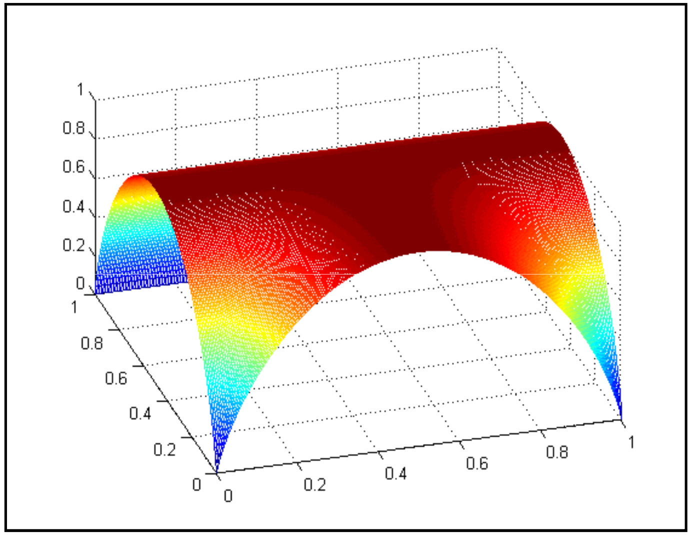
Alternating concavification algorithm



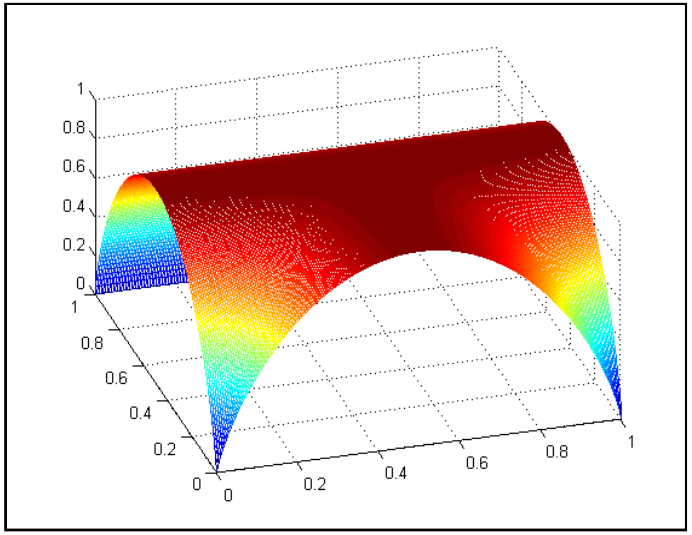
$t = 0$



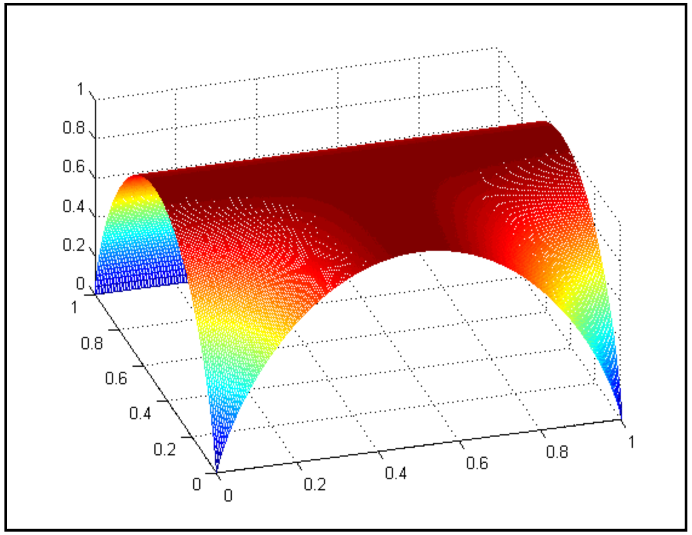
$t = 1$



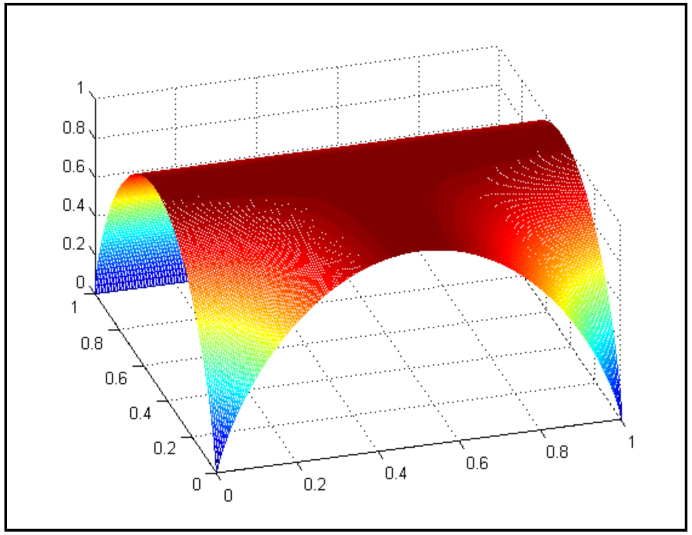
$t = 2$



$t = 3$

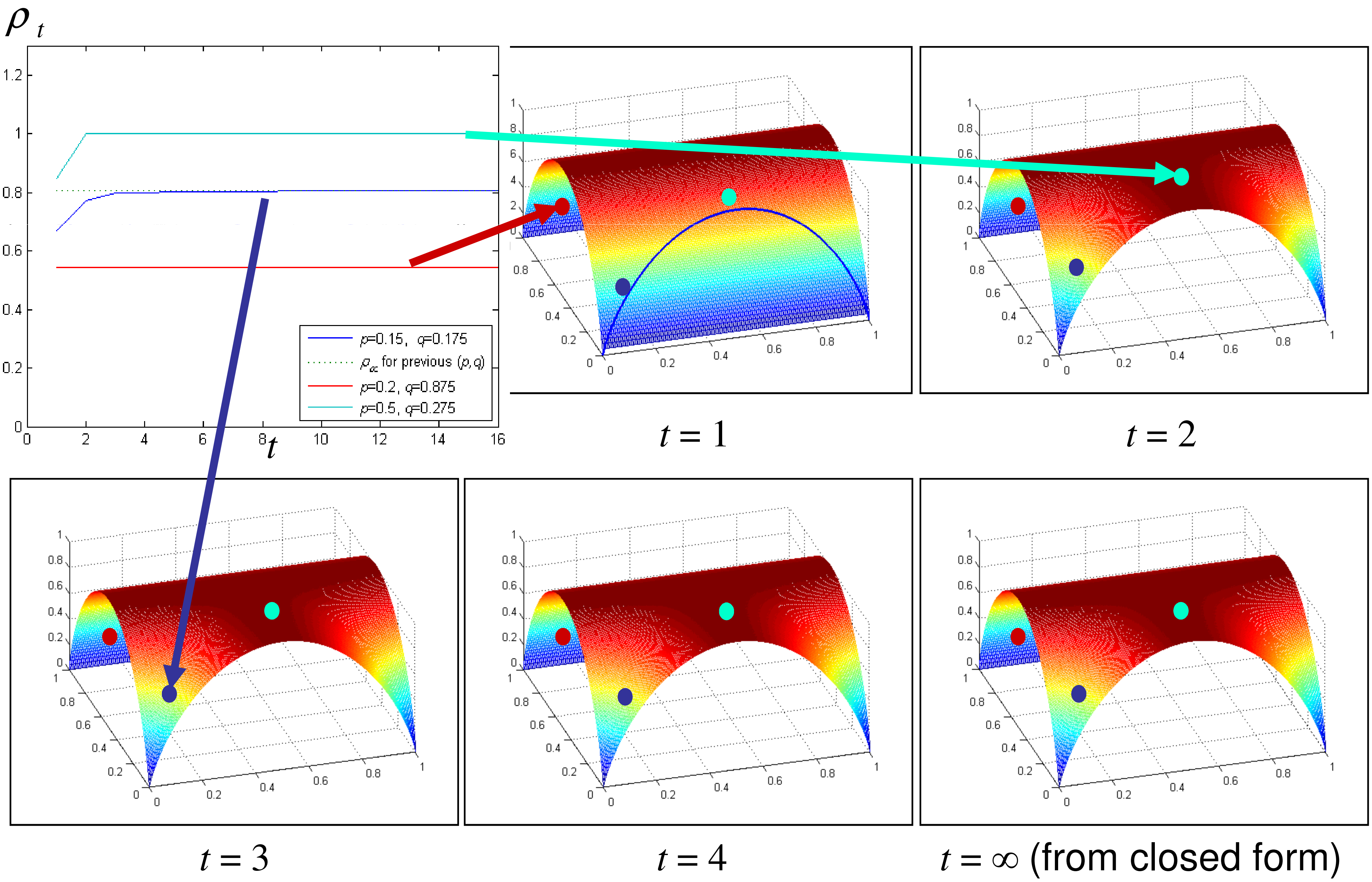


$t = 4$



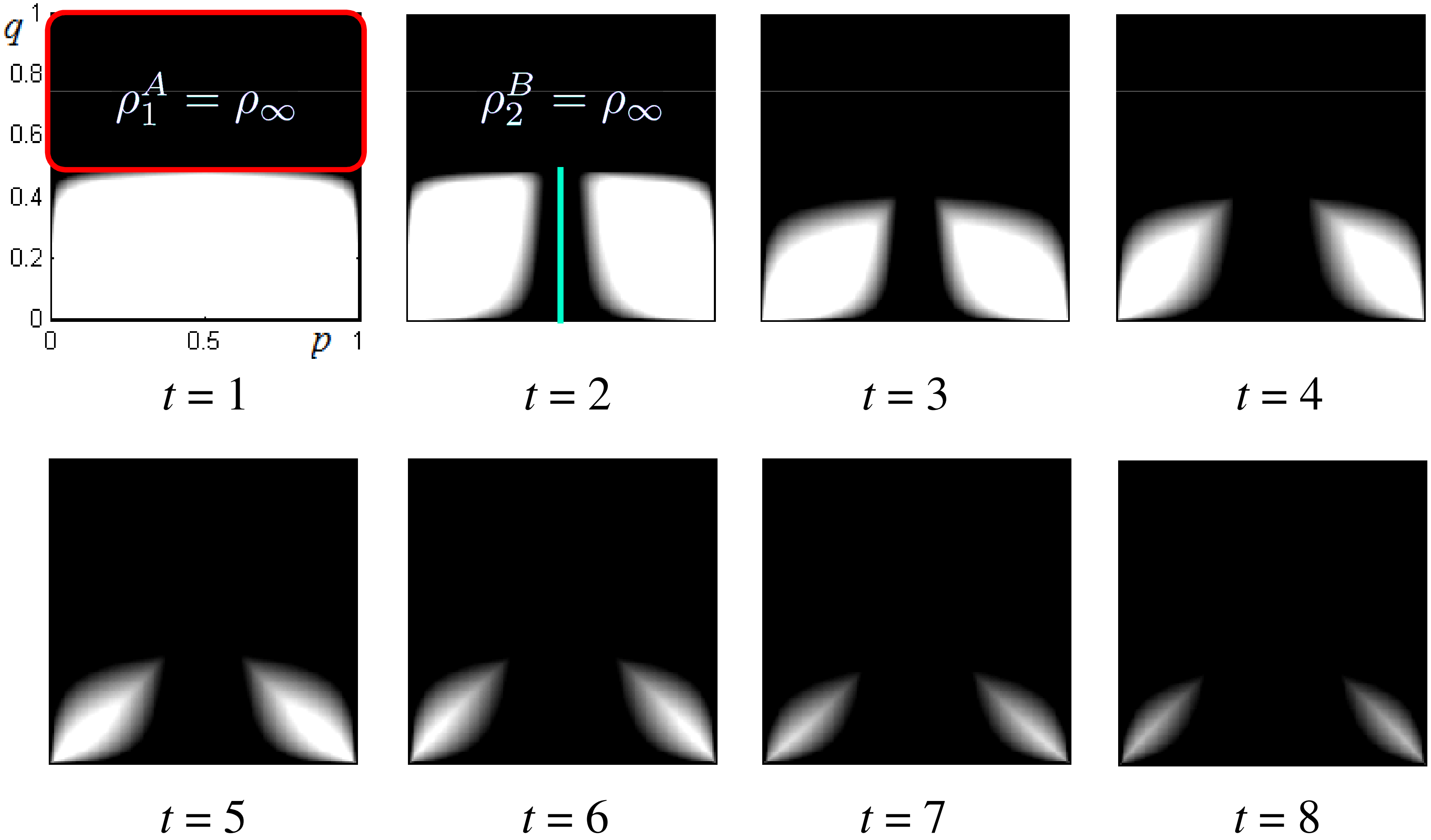
$t = \infty$ (from closed form)

How the surfaces evolve



How the surfaces evolve

Brightness: $|\rho_t(p,q) - \rho_\infty(p,q)|$, black: $< 10^{-4}$



General p_{XY} and f_A, f_B with distortions D_A, D_B

"Limit-free" characterization of $\rho_\infty(p_{XY}, D_A, D_B)$ [Ma, Ishwar: arXiv Oct'09]

ρ_∞ is the least element of $\mathcal{F}$, where

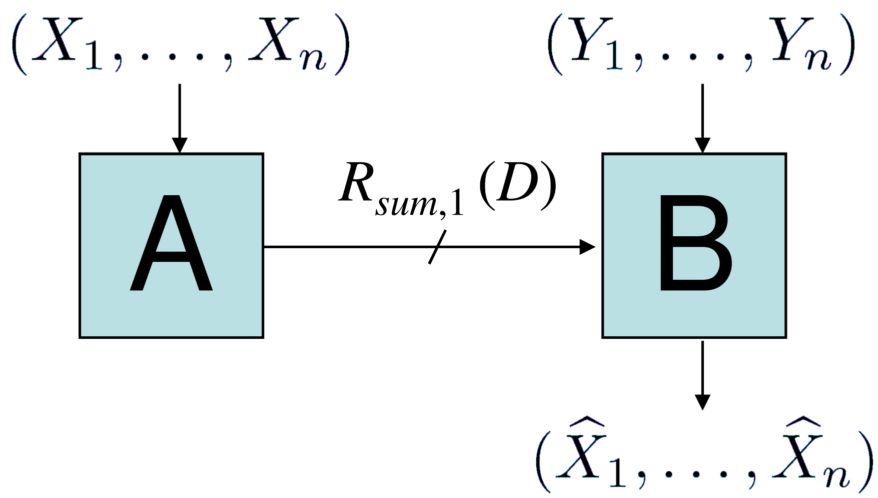
$$\mathcal{F} := \left\{ \rho(p_{XY}, D_A, D_B) \;\middle|\; \begin{array}{l} 1.\ \ \rho \geq \rho_0 \\ 2.\ \text{For all } p_{Y|X},\ \rho(p_X p_{Y|X}, D_A, D_B) \text{ is concave w.r.t. } (p_X, D_A, D_B) \\ 3.\ \text{For all } p_{X|Y},\ \rho(p_Y p_{X|Y}, D_A, D_B) \text{ is concave w.r.t. } (p_Y, D_A, D_B) \end{array} \right\}$$

$$\rho_0 \xrightarrow[\text{Fix } p_{Y|X}]{\text{Concavify wrt } (p_X, D_A, D_B)} \rho_1^A \xrightarrow[\text{Fix } p_{X|Y}]{\text{Concavify wrt } (p_Y, D_A, D_B)} \rho_2^B \longrightarrow \cdots$$

Key question:

For lossy source reproduction, can two messages strictly improve the Wyner-Ziv R-D function? [Ma, Ishwar: ISIT'10]

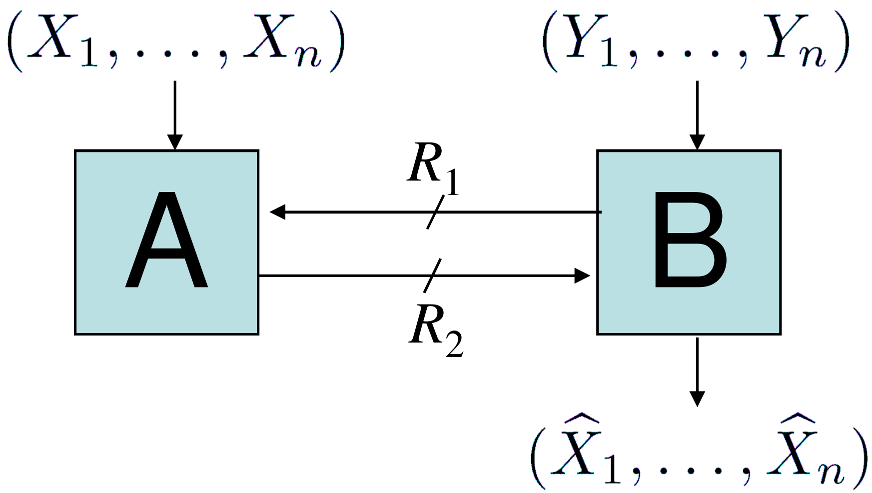
Wyner-Ziv problem



- n samples $(X_i, Y_i) \sim \text{iid } p_{XY}$
- Per-sample distortion measure $d(x, \hat{x})$
- Wyner-Ziv rate-distortion function [Wyner & Ziv IT'76]:

$$\boxed{R^A_{sum,1}(D) = \min_{\substack{U-X-Y \\ \widehat{X}=g(U,Y) \\ E[d(X,\widehat{X})] \le D}} I(X;U|Y)}$$

Kaspi's 2-way src coding problem (simplified version)

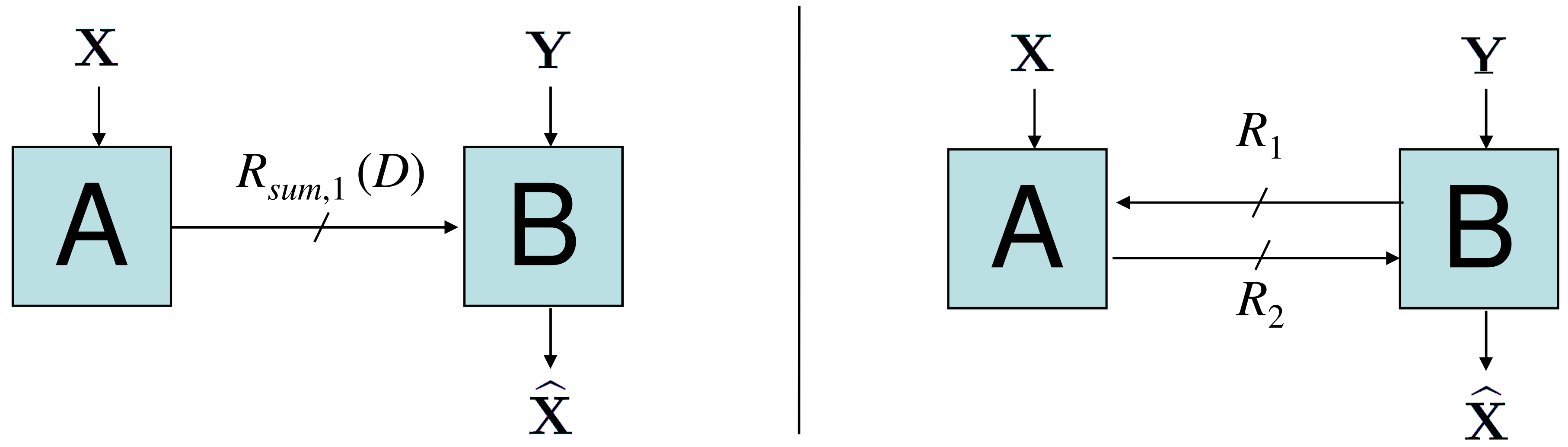


- Same objective: lossy source reproduction
- Two-message interaction [Kaspi IT'85] :

$$R^B_{sum,2}(D) = \min_{\substack{V_1 - Y - X \\ V_2 - (XV_1) - Y \\ \widehat{X} = g(V_1, V_2, Y) \\ E[d(X,\widehat{X})] \le D}} \{I(Y; V_1|X) + I(X; V_2|YV_1)\}$$

Main question

- One message v.s. two messages

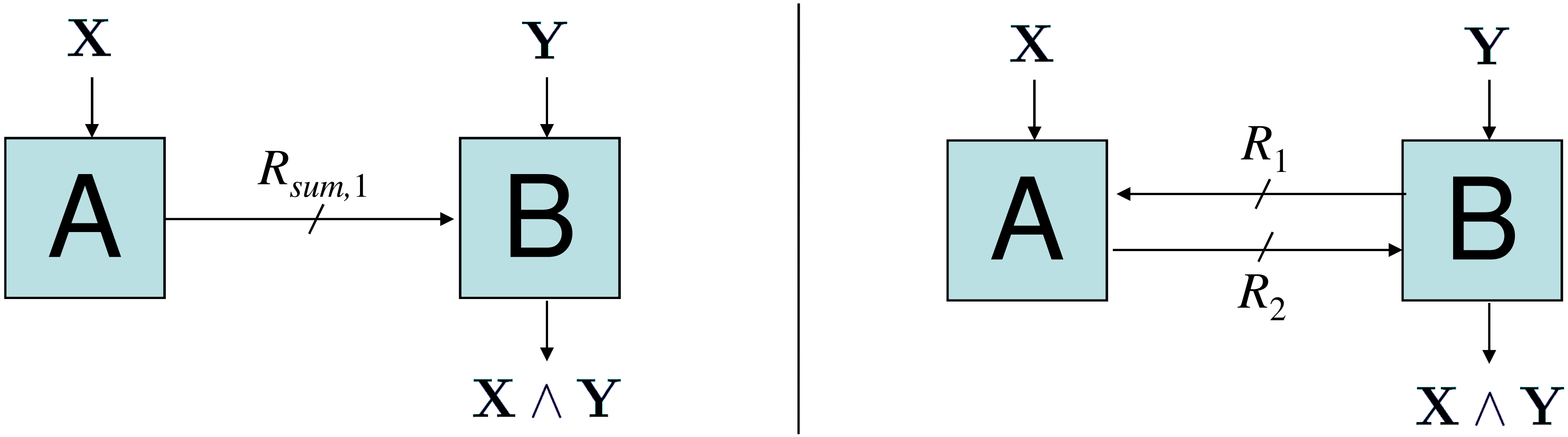


- $R_{sum,1} \geq R_{sum,2}$ always holds
- Question [Kaspi IT'85]: Is interaction useful?

$R_{sum,1} = R_{sum,2}$ or $R_{sum,1} > R_{sum,2}$ for some D?

Related results

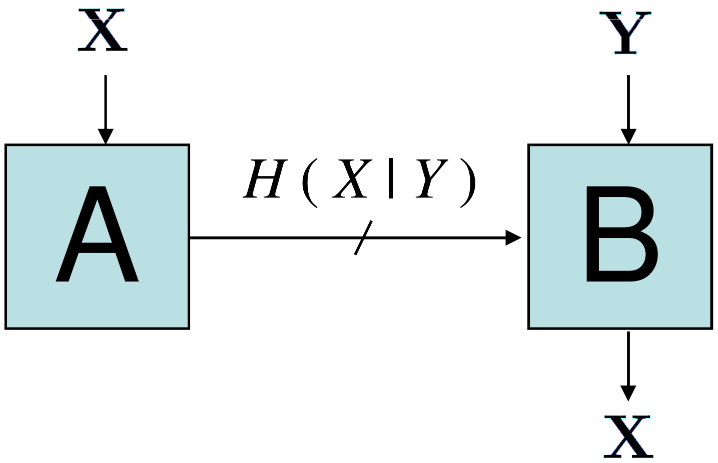
- Lossless function computation [Orlitsky & Roche IT'01], [Ma & Ishwar ISIT'08]
  - $X \perp\!\!\!\perp Y, X \sim \mathrm{Ber}(p), Y \sim \mathrm{Ber}(q)$
  - $f_A(X,Y) = 0, \quad f_B(X,Y) = X \wedge Y$



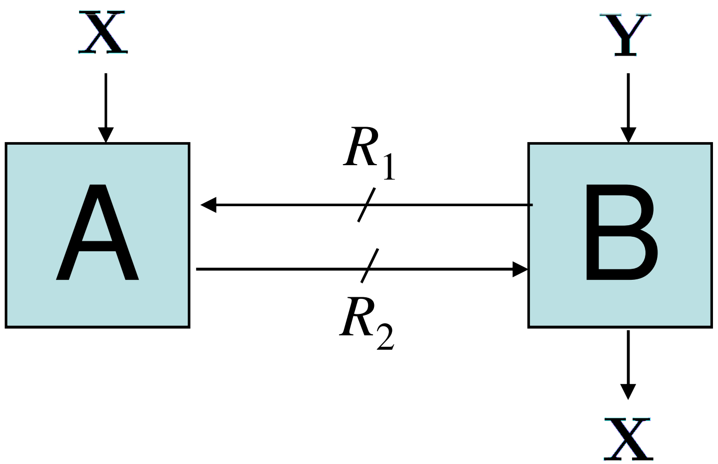
  - Interaction may help: $R_1 + R_2 < R_{sum,1}$
  - $R_{sum,1} / (R_1 + R_2)$ can be arbitrarily large

Related results

- Lossless source reproduction [Slepian & Wolf IT'73] and cutset bound



$R_{sum,1} = H(X|Y)$



even if $R_1 = H(Y)$, $R_2 \geq H(X|Y)$

  – No benefit in using two messages
  – Caveat: interaction may help for nonergodic sources [Yang & He, ISIT'08]

Contribution

- Main question:
  - Lossy source reproduction: $R_{sum,1} = R_{sum,2}$ or $R_{sum,1} > R_{sum,2}$?
- Answer: $R_{sum,1} > R_{sum,2}$ ---- *interaction is useful*
  - Will first show this without explicitly constructing a two-msg scheme
  - Will then show explicit construction in which

    1) $R_{sum,1} / R_{sum,2}$ can be arbitrarily large and simultaneously,

    2) R_1 / R_2 can be arbitrarily small where $R_{sum,1} > R_1 + R_2 \geq R_{sum,2}$
- Key tool: rate reduction functional

Rate reduction functional

"Limit-free" characterization of $\rho_\infty(p_{XY}, D_A, D_B)$ [Ma, Ishwar: arXiv Oct'09]

ρ_∞ is the least element of $\mathcal{F}$, where

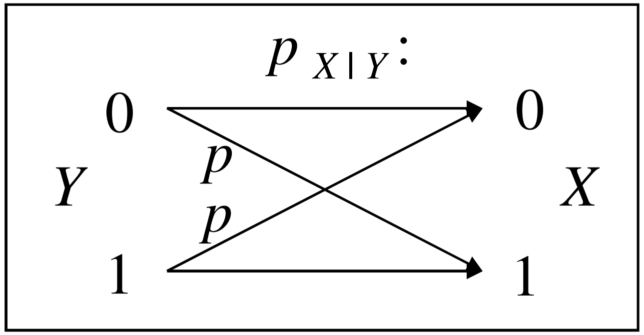
$$\mathcal{F} := \left\{ \rho(p_{XY}, D_A, D_B) \;\middle|\; \begin{array}{l} 1.\ \rho \geq \rho_0 \\ 2.\ \text{For all } p_{Y|X},\ \rho(p_X p_{Y|X}, D_A, D_B) \text{ is concave w.r.t. } (p_X, D_A, D_B) \\ 3.\ \text{For all } p_{X|Y},\ \rho(p_Y\, p_{X|Y}, D_A, D_B) \text{ is concave w.r.t. } (p_Y, D_A, D_B) \end{array} \right\}$$

- Construct example where ρ_1 is not concave w.r.t. p_Y
  - Therefore $\rho_1 \neq \rho_\infty$
  - Can be improved by concavification (using two messages)

$\rho_1(p_{X|Y} p_Y, D)$ is not concave w.r.t. p_Y

- Let $p_{Y_1} \sim Ber(q)$ and $p_{Y_2} \sim Ber(1-q)$

- Let $p_{X|Y} \sim BSC(p) =$



- Let d = binary erasure distortion =

$d(x, \hat{x})$			
$x \backslash \hat{x}$	0	e	1
0	0	1	∞
1	∞	1	0

- We can prove that there exist (p, q, D) such that

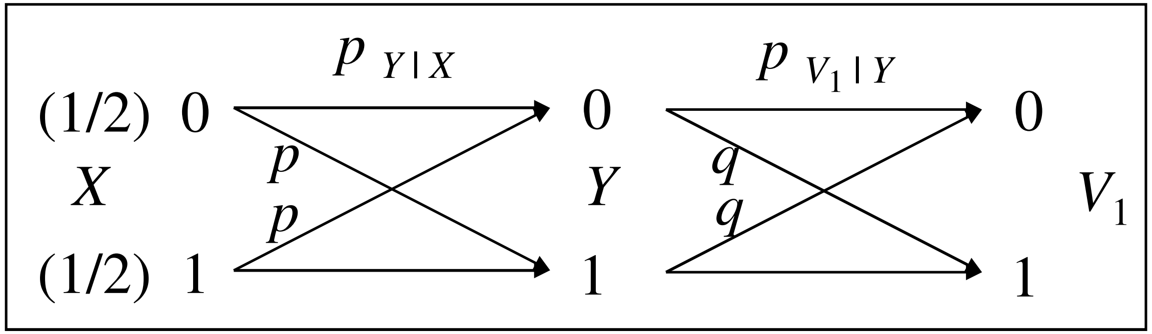
$$\rho_1\left(p_{X|Y}\frac{p_{Y_1}+p_{Y_2}}{2}, D\right) < \frac{1}{2}\rho_1(p_{X|Y}p_{Y_1}, D) + \frac{1}{2}\rho_1(p_{X|Y}p_{Y_2}, D)$$

(would have been $\geq$ if concave)

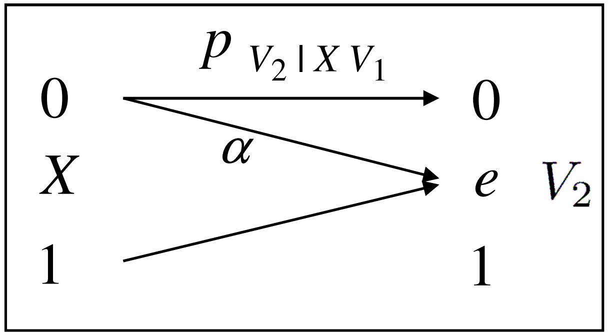
Explicit construction of 2-message aux.r.v.'s

$$R^B_{sum,2}(D) = \min_{\substack{V_1 - Y - X \\ V_2 - (XV_1) - Y \\ \widehat{X} = g(V_1, V_2, Y) \\ E[d(X,\widehat{X})] \le D}} \{ I(Y;V_1|X) + I(X;V_2|YV_1) \}$$

- Let (X, Y) ~ DSBS (p) and
- Choose V_1 to be:

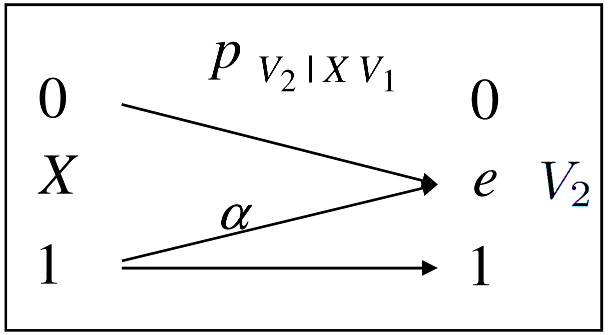


- Choose V_2 as follows:
  - Given $V_1 = 0$



$\widehat{X} = V_2$

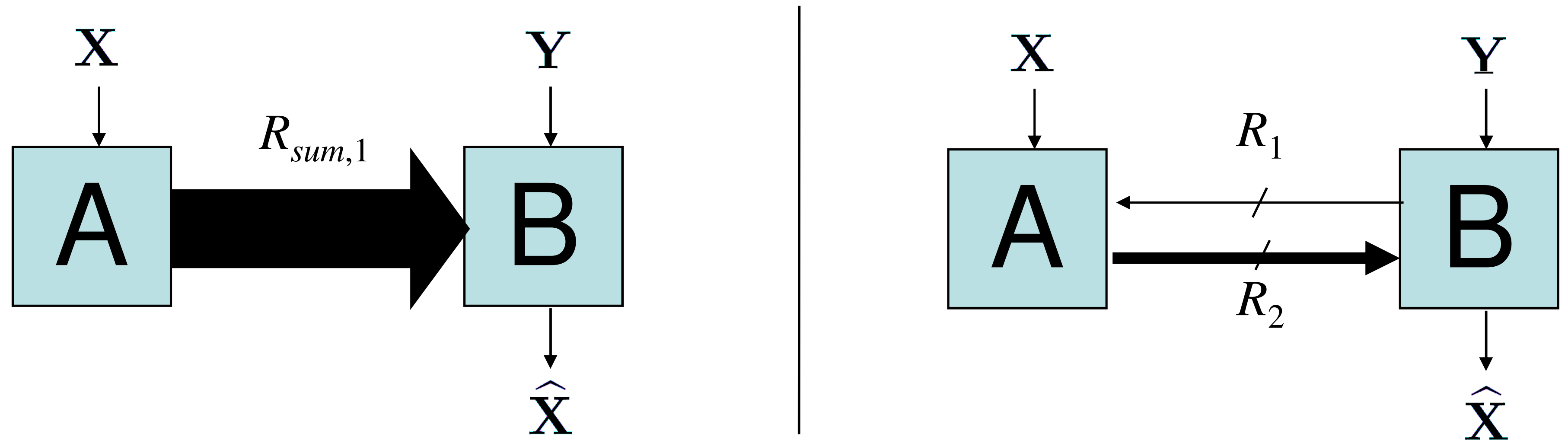
  - Given $V_1 = 1$



$\widehat{X} = V_2$

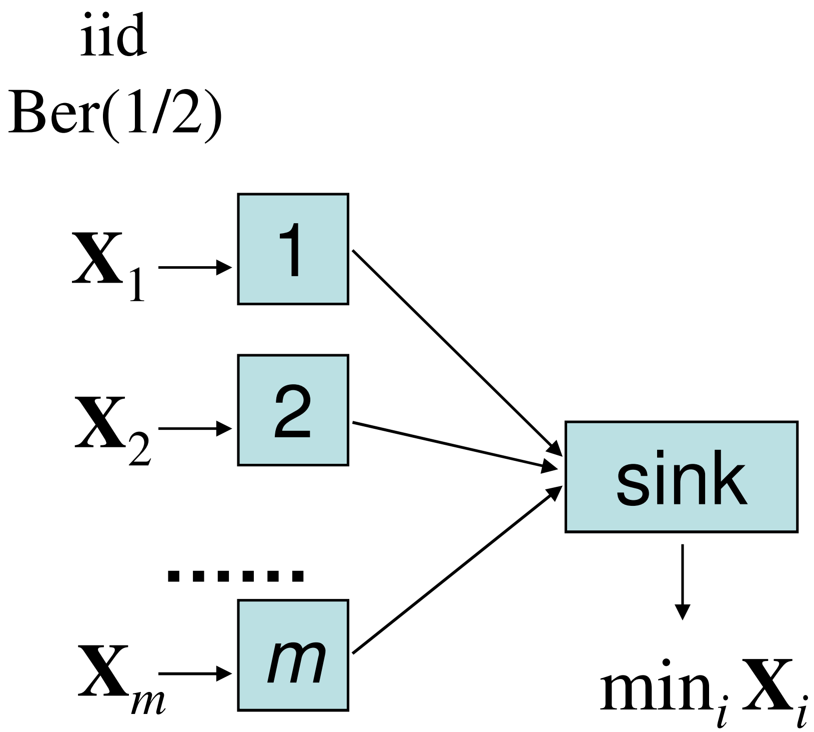
Explicit construction of 2-message aux.r.v.

- When $0 < q \ll 1, 0 < p \ll 1, \quad R_1 \ll R_2 \ll R_{sum,1} \ll 1$

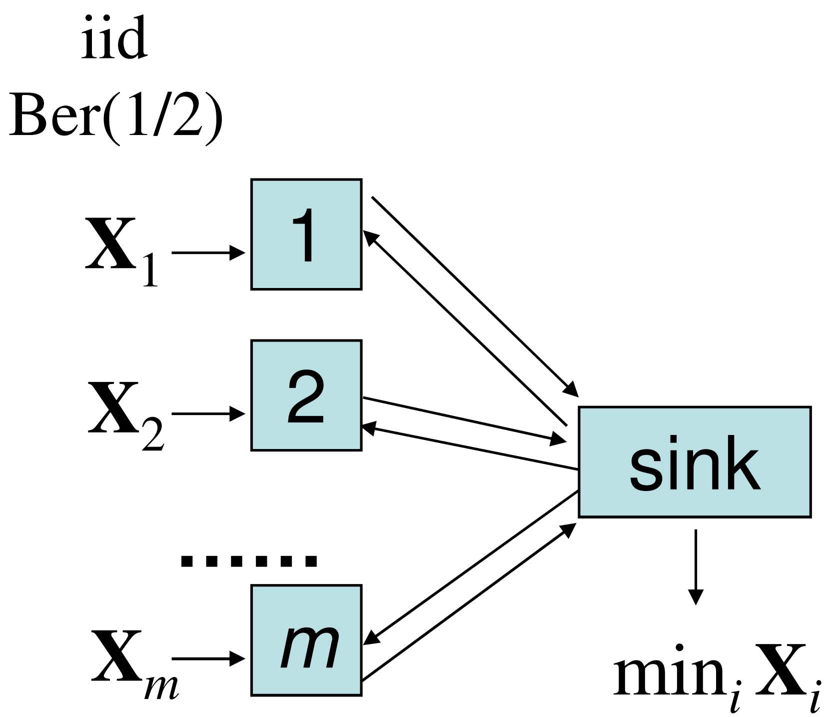


  - $R_{sum,1} / (R_1 + R_2)$ can be arbitrarily large and simultaneously
  - R_1 / R_2 can be arbitrarily small

Interaction changes the scaling law in star networks



$$R_{sum}(m) = m$$



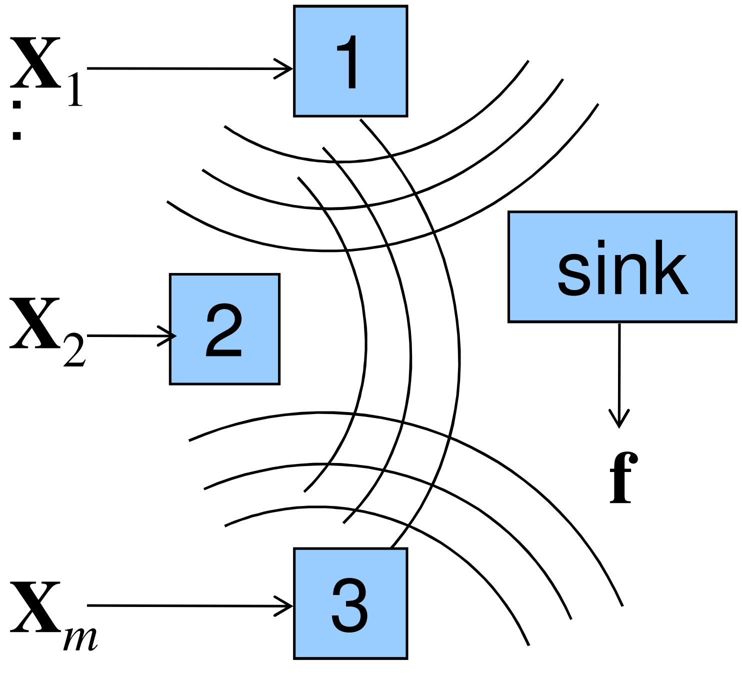
$$1 \leq R_{sum}(m) < 6$$

[submitted to IT; arXiv Nov'08]

Collocated (broadcast) networks

- General independent sources, arbitrary function:
  - Functional convex-geometric characterization for $R_{sum,\infty}$ [ISIT'10]

- Bernoulli sources, symmetric function: [ISIT'09]
  - New upper/lower bounds for $R_{sum,\infty}$
  - New bounds: order-wise tight
  - Cut-set bound: order-wise loose



Concluding remarks

- Interaction is powerful:
  - May get arbitrarily large improvement for lossy source reproduction
  - May get arbitrarily large improvement for function computation
  - May change the scaling law for network computation

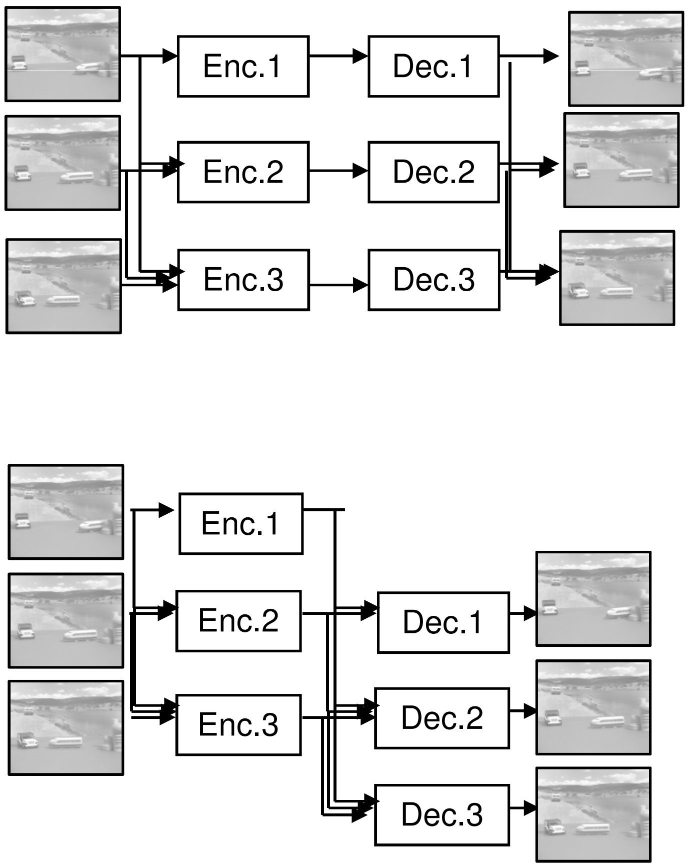
- Characterizing the ultimate limits of interaction:
  - New type of functional single-letter characterization of $R_{sum,\infty}$
  - Alternating "concavification" algorithm for computing $R_{sum,\infty}$
  - New bounds can capture correct scaling behavior in some types of large networks

Future directions

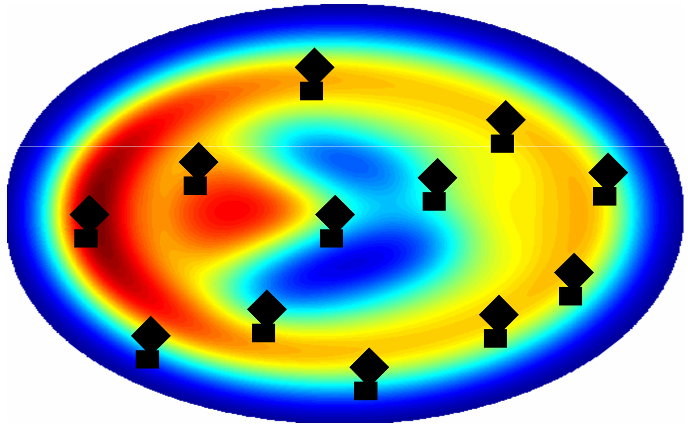
- Interactive (cyclic) network coding
- Interactive function computation in general networks
- Practical interactive code designs
- Interactive computation over noisy channels
- Infinite messages with infinitesimal rate => "Calculus" for source coding?

Other research topics

Delayed sequential coding of video frames



Field reconstruction using noisy one-bit sensors



Modulation formats in 40Gbps fiber-optic communication systems